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Classification of Razor Blades to the filtration equation—the sublinear case

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Abstract

We study nonnegative solutions of the filtration equation $u_t = \Delta\varphi(u)$ in $\mathbb{R}^N(N \geqslant 2)$, where φ is continuous, increasing and sublinear. More precisely, we study the Razor Blades, i.e., solutions which may be singular at $|x| = 0$ for $t > 0$, and start with zero initial data. We first prove a nonexistence result when φ is too sublinear and we show an axial trace result in the other case: there exist a time $\tau \equiv \tau(u)$ and a Radon measure λ on $[0, \tau)$ such that

$$u_t = \Delta\varphi(u) + \delta_0 \otimes \lambda(t) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times [0, \tau)).$$

Then u has a strong singularity on $|x| = 0$ for any $t > \tau$. We then prove existence of such solutions for any λ and τ as above, and give a uniqueness result for those solutions. Finally, we make a complete study of self-similar solutions (in the power case) which classify the possible asymptotic behaviours.

Keywords: Fast-diffusion equation; Sublinear filtration; Well-posed problem; Borel measures; Singular solutions

1. Introduction

In this paper, we intend to give a complete classification of the nonnegative solutions to the sublinear filtration equation

$$u_t = \Delta\varphi(u) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N_* \times (0, \infty)), \tag{1.1}$$

where $N \geqslant 2$. More precisely, we assume that $u(x, 0) = 0$ for $x \neq 0$, but allow infinite values of u on the t-axis $\{|x| = 0\}$ and want to classify the different solutions which

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may exist. Let us first recall that there is an important literature concerning similar investigations for elliptic and parabolic equations like

$$\text{(E)} \quad -\Delta u \pm u^q = 0, \qquad \text{(P)} \quad u_t - \Delta u + u^q = 0.$$

For the elliptic equation (E), these works were initiated first by Serrin [31] (see also [17,22,33]). We refer to the book of Véron [34] for a complete overview of such results, as well as further references. Concerning parabolic equations, let us mention the works of Brezis–Friedman [5] and Kamin–Peletier [19,20], where singularities are considered in the initial data. In the case of the heat equation with absorption (P), existence of a so-called *Very Singular Solution* (*VSS*) was discovered by Brezis et al. [6] and Galaktionov et al. [16]. This special solution, which may be viewed as a classical solution with initial data "$+\infty \cdot \delta_0$" plays an important role in describing singular solutions and long-time behaviour. Then, generalizations of the VSS were obtained for porous medium and p-Laplace equations [26,27], respectively:

$$\text{(PME)} \quad u_t - \Delta u^m + u^q = 0,$$

$$\text{(PLE)} \quad u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^q = 0,$$

and a complete classification of singular solutions for (PME) was performed by Kamin et al. in [21].

More recently, Marcus and Véron introduced the concept of *Trace* for (E) and (P), which allows for a complete classification of solutions by means of generalized *Borel measures* [23–25], and the author extended their results to (PME) in [8,9] (and with Vazquez in [13]). Let us recall that a nonnegative Borel measure v may be written as $v = (\mathscr{S}, \mu)$, where v blows-up on some compact set $\mathscr{S} \subset \mathbb{R}^N$ and μ is a Radon measure on the regular set $\mathscr{R} = \mathbb{R}^N \backslash \mathscr{S}$. Thus, in the case $v = +\infty\delta_0$, we have the decomposition $\mathscr{S} = \{0\}$, $\mu = 0$.

In the case of the so-called *fast-diffusion* equation

$$u_t = \Delta u^m, \quad 0 < m < 1, \tag{1.2}$$

existence of solutions with singular initial data was shown by Brezis–Friedman [5], and Pierre [29] showed that the initial trace of distributional solutions is necessarily locally bounded (i.e., it is a Radon measure). Hence, general Borel measures such as "$+\infty \cdot \delta_0$" are not allowed in this setting. However, Vazquez and Véron [32] obtained a singular solution in the range $m_c < m < 1$, $m_c = (N-2)/N$, which has the self-similar form

$$U(x,t) = \left(\frac{Ct}{|x|^2}\right)^{\frac{1}{1-m}}, \quad C > 0. \tag{1.3}$$

This solution was named Razor Blade in [32] referring to the way the singularity advances along the t-axis. We will use the same terminology here, but in a somewhat wider sense (see definition below). The particularity of this solution is reflected in the fact that the constant singularity at $x = 0$ is *strong*, i.e., U is never locally integrable near $x = 0$:

$$\forall r, t > 0, \quad \int_{B_r(0)} U(x,t) = +\infty.$$

Hence U is not a distributional solution, although it is a classical solution for $x \neq 0$. As for equation (P), this special solution is very important to describe more general singularities and the asymptotic behaviour of solutions. In this spirit, the author and J.L. Vazquez extended the whole theory of weak solutions of (1.2) to take into account solutions like (1.3), which was named *infinite point-source solution* (IPSS) in [11,12]. The nondistributional theory developed there is complete since we showed well-posedness of the Cauchy-problem with initial data any Borel measure, but only strong singularities are allowed for $t > 0$. Thus, this theory leaves out solutions with *weak singularities*, i.e., integrable singularities of u, like in the following example (see for instance [30]):

$$\begin{cases} u_t - \Delta u^m = \delta_0(x) \otimes 1(t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4}$$

It is the purpose of this paper to give a complete description of all singular solutions which start with zero initial value for $x \neq 0$ and which may be singular on the t-axis, that is on the set $\{x = 0, \ t \geqslant 0\}$. Thus, we include solutions like (1.3) and (1.4) in order to complete the works initiated in [11,12]. We also complete previous studies of singularities for parabolic equations in which singularities were only supported at time $t = 0$. Moreover, we will perform the analysis for the more general filtration equation (1.1).

The filtration function φ is assumed here to be sublinear. The analysis for superlinear φ's will be performed in [10]. Thus, we assume that $0 \leqslant \varphi(u) \leqslant Cu^m$ for some $C > 0$ and $m < 1$. However, it is well-known at least in the case $\varphi(u) = u^m$, that if m is too small, namely

$$0 < m \leqslant m_c = \frac{N-2}{N},$$

then no fundamental solution exists, whereas they always exist if $m > m_c$. This phenomenon is due to the fact that below m_c, if the initial data is too singular, then it does not diffuse (see [5,29]). Technically speaking, this is also a consequence of the lack of regularizing effect from L^1 into L^∞ when $m \leqslant m_c$.

We will thus make an extensive use of the works of Dahlberg and Kenig [15] who studied the weak solutions of

$$u_t = \Delta\varphi(u) \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.5}$$

under the following assumptions: the filtration function φ is continuous in $[0,\infty)$, with $\varphi(0) = 0$, φ is derivable in $(0,\infty)$ and there exists two constants γ, μ such that

$$m_c < \gamma \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant \mu < 1.$$

We will follow the same assumptions in this paper, and call $\mathscr{L}^*_{\text{sub}}$ the class of such filtration functions φ that are concave, but we will study a different class of solutions: we consider a function $u \geqslant 0$ in $\bar{Q} = \mathbb{R}^N \times [0, \infty)$ such that

(H1) $u \in \mathscr{C}^0(\bar{Q}; \mathbb{R}_+ \cup \{+\infty\})$.
(H2) $u_t = \Delta\varphi(u)$ in $\mathfrak{D}'(\mathbb{R}^N_* \times (0, \infty))$.
(H3) $u(x, 0) = 0$ for $x \neq 0$.

If u is continuous in $\mathbb{R}^N \times (0, \infty)$, then the works of [15] prove that necessarily, either $u \equiv 0$, or u is a fundamental solution with mass $c > 0$. In any other case, we will call u a Razor Blade.

Definition 1. A Razor Blade is a function $u \geqslant 0$ satisfying (H1)–(H3) which is not zero and not a fundamental solution. In particular, u is not bounded near $x = 0$, at least for some $t > 0$.

We can always assume that

$$\sigma(u) = \inf\{t > 0 \mid u(x, t) > 0 \text{ for some } x \in \mathbb{R}^N\} = 0,$$

thus leaving out the possibility that u be identically zero on some time interval $(0, t)$. The case $\sigma(u) > 0$ can be treated by initializing the problem at $t = \sigma(u)$. Note also that a Razor Blade u could become continuous in the whole space $\mathbb{R}^N$ after some time $s > 0$, thus becoming a weak solution on $\mathbb{R}^N \times (s, \infty)$. As we shall see, in some sense, any Razor Blade is of type (1.4) on some interval $(0, \tau)$ (with a measure instead of $1(t)$), and then of type (1.3) for $t > \tau$. Let us now mention our main results:

(A) *Nonexistence when* $0 \leqslant \gamma \leqslant \mu \leqslant m_c$. In this range, we prove that no singular solution can exist. This is a consequence of the fact that isolated points have zero capacity in $W^{2,1/(1-m_c)}$, hence the t-axis is not "seen" by the equation. This result was already proved in [11] in the power case. Thus, in the rest of the paper we shall assume that $m_c < \gamma \leqslant \mu < 1$, and we note $\mathscr{L}^*_{\text{sub}}$ the class of such filtration functions.

(B) *Infinite point-source solutions*. This kind of solutions have only strong singularities at $x = 0$ for any $t > 0$, and they only exist for $\varphi \in \mathscr{L}^*_{\text{sub}}$. In the case $\varphi(u) = u^m$, existence and uniqueness of the IPSS was obtained in [11] after separation of variables. For general φ's, existence is easily obtained by taking limits of fundamental solutions, but separation of variables is not available to us. Thus, we use another proof to show uniqueness of the IPSS.

(C) *Existence of an axial trace*. If $\varphi \in \mathscr{L}^*_{\text{sub}}$, we prove that the behaviour of singular solutions near the t-axis may be described as follows: for any singular solution u, there exists a time $\tau(u) \in [0, \infty]$ and a measure $\lambda \in \mathscr{M}^+([0, \tau))$ such that

$$u_t = \Delta\varphi(u) + \delta_0(x) \otimes \lambda(t) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times [0, \tau)),$$

while u has a standing strong singularity on the axis for any $t > \tau$:

$$\forall t > \tau, \quad \forall r > 0, \quad \int_{|x|<r} u(x,t)\, dt = +\infty.$$

Thus, we may describe the "trace" of u on the axis by $\mathrm{tr}_{|x|=0}(u) = (\lambda, \tau)$, also noting $\mathscr{S} = [\tau, \infty)$ (the singular set). The possible presence of initial mass placed at $x = 0$ is then included in λ: it is exactly $\lambda(0)$. The techniques involved here are rather close to the ones that were used by different authors for elliptic equations.

(D) *Existence of Razor Blades*. We prove existence of singular solutions possessing an arbitrary axial trace (λ, τ) for $\varphi \in \mathscr{L}^*_{\mathrm{sub}}$. This is achieved by considering the equation with a forcing term and passing to the limit when the forcing term converges weakly to the axial measure.

(E) *Uniqueness of Razor Blades*. Our uniqueness result is based upon the works of Dahlberg and Kenig [15], through the use of a minimal solution with a given axial trace. Thus we prove that any Razor Blade is uniquely determined by its axial trace (λ, τ), which gives the complete classification of such solutions.

(F) *Self-similarity and asymptotic behaviour.* In the power case $\varphi(u) = u^m$, with $m_c < m < 1$ and $(\lambda(t), \tau) = (t^\sigma, +\infty)$ the Razor Blade has the special form:

$$u(x,t) = t^\alpha f(xt^{-\beta}),$$

where f satisfies an elliptic equation. Note that in this case, the IPSS is given by (1.3). From the behaviour of f at $x = 0$, we deduce the following asymptotics:

If $\sigma < m/(1-m)$, u behaves like $t^{\sigma/m}$ when $t \to +\infty$.

If $\sigma = m/(1-m)$, u behaves like $t^{m/(1-m)}$ when $t \to +\infty$, but not like (1.3).

If $\sigma > m/(1-m)$, u behaves like (1.3) when $t \to +\infty$.

On the other hand, in presence of strong singularities, i.e., if $\tau < \infty$, then u behaves like (1.3) when $t \to +\infty$.

The paper ends with a comments section and an appendix.

2. Preliminaries

2.1. Notations and assumptions

We consider the class $\mathscr{L}_{\mathrm{sub}}$ of sublinear filtration functions φ satisfying: $\varphi \in \mathscr{C}^0([0, \infty))$, $\varphi(0) = 0$, φ' exists everywhere on $(0, \infty)$, $\varphi'(u) > 0$ for $u > 0$ and such that for some γ and μ,

$$0 < \gamma \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant \mu < 1. \tag{2.1}$$

Moreover, we require the normalization condition $\varphi(1) = 1$. As we pointed out in the Introduction, the critical exponent $m_c = (N-2)_+/N$ plays a crucial role in the existence of solutions. Thus we are led to consider the following subclass $\mathscr{L}^*_{\text{sub}}$ of filtration functions:

Definition 2. The subclass $\mathscr{L}^*_{\text{sub}}$ consists of concave filtration functions $\varphi \in \mathscr{L}_{\text{sub}}$ satisfying

$$m_c < \gamma \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant \mu < 1. \tag{2.2}$$

As a consequence, there exists some constant $v > 0$ such that

$$u^\gamma \leqslant \varphi(u) \leqslant u^\mu \quad \text{for } u \geqslant 1, \tag{2.3}$$

$$0 \leqslant \varphi(u) \leqslant u^v \quad \text{for } 0 < u < \infty, \tag{2.4}$$

$$0 \leqslant \varphi(u) \leqslant u^\beta \quad \text{for } 0 < u < 1, \tag{2.5}$$

where $\beta = \min\{\mu, v\}$. Moreover,

$$\gamma u^{\gamma-1} \leqslant \varphi'(u) \leqslant \mu u^{\mu-1} \quad \text{for } u \geqslant 1. \tag{2.6}$$

Let us recall (see Definition 1) that a Razor Blade u is a nonnegative function which is not identically zero, and not a fundamental solution of (1.5), satisfying

(H1) $u \in C^0(\bar{Q}; \mathbb{R}_+ \cup \{+\infty\})$.
(H2) $u_t = \Delta\varphi(u)$ *in* $\mathfrak{D}'(\mathbb{R}^N_* \times (0, \infty))$.
(H3) $u(x, 0) = 0$ *for* $x \neq 0$.

Any Razor Blade u may be considered as a solution of

$$u_t = \Delta\varphi(u) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N_* \times [0, \infty)),$$

since we assume that $u(x, 0) = 0$ in the continuous sense for $|x| \neq 0$.

2.2. Weak solutions of the nonsingular problem

We refer to [15] for proofs of the following facts concerning the filtration equation:

$$u_t = \Delta\varphi(u) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times (0, \infty)).$$

Let us first recall the local L^1- and L^∞-estimates of Dahlberg and Kenig: H_φ is the function

$$H_\varphi(s) = \begin{cases} 1 & \text{if } 0 < s < 1, \\ s^p[\varphi(s)/s]^{N/2} & \text{if } s > 1, \end{cases}$$

where $p \geqslant 1$ is a constant depending only on N and γ, μ, and

$$\psi_{x_0,R} = \psi\Big(\frac{x-x_0}{R}\Big)R^{-N}, \quad \psi \in \mathscr{C}_0^\infty(\mathbb{R}^N), \quad 0 \leqslant \psi \leqslant 1,$$

$\psi = 1$ in B_R and $\psi = 0$ in $\{|x| > 2R\}$. For any $\theta > 0$, we note

$$C_\theta(\psi) = \left[\int_{\mathbb{R}^N} |\Delta\psi|^{\frac{1}{1-\theta}} \psi^{-\frac{\theta}{1-\theta}}\right]^{1-\theta},$$

and there exists a $\psi = \bar\psi$ such that $C_\beta(\bar\psi), C_\mu(\bar\psi) < \infty$, where $\beta = \min\{\mu, v\}$. Note that $C_\theta(\bar\psi_{x_0,R}) = R^{-2}C_\theta(\bar\psi)$. Then we have:

Theorem 2.1 (Dahlberg–Kenig). *Let $\varphi \in \mathscr{L}_{\text{sub}}$. Then for any continuous weak solution u of $u_t = \Delta\varphi(u)$ in $B_{2R}(x_0) \times [s,t]$, there exists two constants C_1, C_2 such that if*

$$\max\left\{\int_{\mathbb{R}^N} u(x,t)\bar\psi_{x_0,R}(x)\,dx; \int_{\mathbb{R}^N} u(x,s)\bar\psi_{x_0,R}(x)\,dx\right\} \leqslant C_1,$$

$$\left|\left[\int_{\mathbb{R}^N} u(x,t)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta} - \left[\int_{\mathbb{R}^N} u(x,s)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\beta}\right| \leqslant C_2|t-s|/R^2.$$

If the max *is greater than C_1, then*

$$\left|\left[\int_{\mathbb{R}^N} u(x,t)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu} - \left[\int_{\mathbb{R}^N} u(x,s)\bar\psi_{x_0,R}(x)\,dx\right]^{1-\mu}\right| \leqslant C|t-s|/R^2.$$

If, moreover, we assume that $m_c < \gamma \leqslant \mu < 1$, *that is, $\varphi \in \mathscr{L}^*_{\text{sub}}$, then for any weak solution $u \in \mathscr{C}^0(B_{4R}(x_0) \times [0,T])$, if $t/R^2 > 1$, we have*

$$\sup_{x \in B_R(x_0)} H_\varphi(u(x,t)) \leqslant C\left\{\frac{1}{t^{N/2}}\int_{B_{4R}(x_0)} u(x,0)\,dx + \Gamma_\varphi(t/R^2)\cdot(t/R^2)^{N/2}\right\}. \tag{2.7}$$

In fact, the Cauchy problem is well-posed provided $\varphi \in \mathscr{L}^*_{\text{sub}}$:

Theorem 2.2 (Dahlberg–Kenig). *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$. Then any continuous distributional solution u* of *$u_t = \Delta\varphi(u)$ has a trace at $t = 0$* which *is a Radon measure. Moreover, for any Radon measure μ, there exists one and only one continuous solution u with initial trace μ.*

In particular, we have existence and uniqueness of fundamental solutions, i.e., the solutions v_c with initial data $c\delta_0$.

2.3. Nonexistence of Razor Blades

We begin with the nonexistence of Razor Blades and fundamental solutions when $\varphi \in \mathscr{L}_{\text{sub}}$, with $0<\gamma\leqslant\mu<m_c$. This result extends Lemma 10.2 of [11], and the proof is similar.

Theorem 2.3. *Let $\varphi\in\mathscr{L}_{\text{sub}}$ with $0<\gamma\leqslant\mu<m_c$ and $u\in\mathscr{C}^0(\mathbb{R}^N_*\times[0,\infty))$ such that*

$$u_t = \Delta\varphi(u) \quad in\ \mathfrak{D}'(\mathbb{R}^N_*\times[0,\infty)).$$

Then $u\equiv 0$ a.e. in $\mathbb{R}^N\times[0,\infty)$.

Proof. For $T>0$ arbitrary, we will first prove that $u\in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$. Since $\{0\}$ has zero $C_{2,1/(1-\mu)}$-capacity (because $\mu<m_c$), there exists a sequence $v_n\in\mathscr{C}_0^\infty(\mathbb{R}^N)$ such that $0\leqslant v_n\leqslant 1$, $v_n=1$ on a neighbourhood of $\{0\}$ and $v_n\to 0$ in $W^{2,1/(1-\mu)}(\mathbb{R}^N)$. For $\alpha>\frac{2}{1-\mu}$, and $\psi\in\mathscr{C}_0^\infty(\mathbb{R}^N)$, nonnegative, we use the test function $\zeta_n^\alpha=[\psi(1-v_n)]^\alpha\in\mathscr{C}_0^\infty(\mathbb{R}^N_*)$:

$$\int u(t)\zeta_n^\alpha - \int_0^t\int\varphi(u)\Delta\zeta_n^\alpha = 0, \tag{2.8}$$

since the support of ζ_n is outside $\{0\}$. Then we estimate

$$\int_0^t\int\varphi(u)|\Delta\zeta_n^\alpha|\leqslant C(\zeta_n^\alpha)\left[\int_0^T\int u\zeta_n^\alpha\right]^\mu$$

with

$$C(\zeta_n^\alpha)=T^{1-m}\left[\int|\Delta\zeta_n^\alpha|^{\frac{1}{1-\mu}}\zeta_n^{-\frac{\alpha\mu}{1-\mu}}\right]^{1-\mu}\leqslant C(\mu,N)||\psi||_{W^{2,1/(1-\mu)}(\mathbb{R}^N)}.$$

The estimate of $C(\zeta_n^\alpha)$ comes from easy computations and the fact that $(1-v_n)$ remains bounded in $W^{2,1/(1-\mu)}(\mathbb{R}^N)$. Thus integrating (2.8) on $(0,T)$, one gets

$$\int_0^T\int u\zeta_n^\alpha\leqslant C'\left[\int_0^T\int u\zeta_n^\alpha\right]^\mu. \tag{2.9}$$

Hence it follows that for some constant C'' depending only on μ, N and $||\psi||_{W^{2,1/(1-\mu)}(\mathbb{R}^N)}$,

$$\int_0^T\int u\zeta_n^\alpha\leqslant C''(\mu,N,||\psi||_{W^{2,1/(1-\mu)}(\mathbb{R}^N)}). \tag{2.10}$$

Passing to the limit when n goes to infinity, and using the fact that ψ is arbitrary, we obtain that $u\in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$. Now we take $\psi\in\mathscr{C}_0^\infty(\mathbb{R}^N\times[0,T))$, and put $\zeta_n=$

$\psi(1-v_n)$ as test function, where v_n is as above:

$$\int u(t)\zeta_n - \int_0^t \int u\partial_t\zeta_n - \int_0^t \int \varphi(u)\Delta\zeta_n = 0.$$

When n goes to infinity, the two first terms converge by dominated convergence and for the last one, we use both the fact that $\zeta_n \underset{n\to\infty}{\to} \varphi$ in $W^{2,1/(1-\mu)}(\mathbb{R}^N)$, and that $\varphi(u) \in L^{1/\mu}(0,T;L^{1/\mu}_{\mathrm{loc}}(\mathbb{R}^N))$ (because $\varphi(u) \leqslant C_2 u^\mu$ and (2.10)), so that

$$\int_0^t \int \varphi(u)\Delta\zeta_n \underset{n\to\infty}{\to} \int_0^t \int \varphi(u)\Delta\psi.$$

Hence we obtain that u satisfies

$$\int u(t)\psi - \int_0^t \int u\psi_t - \int_0^t \int \varphi(u)\Delta\psi = 0,$$

which means that $u \in L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap \mathscr{C}^0(\mathbb{R}^N \times [0,T))$ is a solution of

$$u_t - \Delta\varphi(u) = 0 \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times (0,T)),$$

with zero initial data. We have now to prove that $u \equiv 0$. Let us first note that by the estimates of Dahlberg–Kenig (see Theorem 2.1 with $s=0$), one has for some constant $C = C(\mu, T, N) > 0$, and R big enough:

$$\int_{B_R(x_0)} u(x,t)\, dt \leqslant CR^{N-2/(1-\mu)}.$$

Actually, the estimates of Proposition 5.1 are valid for $x_0 \neq 0$, but since the constants are independent of x_0 (and u is locally integrable near x_0), we have also the estimate at $x_0 = 0$. The problem now is that since $\mu \leqslant m_c$, letting R go to infinity does not yield $u(t) = 0$. However, using the same technique as in the proof of [18, Theorem 2.3], one easily obtains that $u \equiv 0$ almost everywhere:

If we set $w(x) = \int_0^t \varphi(u)(x,\sigma)\, d\sigma$, which is defined almost everywhere in $\mathbb{R}^N$, then w is subharmonic (in the sense of distributions). Indeed, if $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, $\psi \geqslant 0$,

$$\langle \Delta w, \psi \rangle = \int_0^t \int_{\mathbb{R}^N} \varphi(u)\Delta\psi = \int_{\mathbb{R}^N} u(t)\psi \geqslant 0,$$

and thus

$$\begin{aligned} w(x_0) &\leqslant \frac{C}{R^N} \int_0^t \int_{B_R(x_0)} w(x)\, dx \\ &\leqslant \frac{C'}{R^N} \int_0^t R^{N(1-\mu)} \left[\int_{B_R(x_0)} u(s) \right]^\mu ds \\ &\leqslant C'' R^{-2\mu/(1-\mu)}, \end{aligned}$$

which goes to zero when R goes to infinity. $\square$

2.4. Some comparison results

The following result is proved in [11, Proposition 1.4] in the case $\varphi(u) = u^m$. The adaptation to φ's satisfying (2.2) requires only minor changes.

Lemma 2.4. *Let $\varphi \in \mathscr{L}^*_{\rm sub}$, Ω a regular open subset of $\mathbb{R}^N$ (not necessarily bounded) and u, v be two weak solutions of*

$$u_t = \Delta\varphi(u) \quad in\ \mathfrak{D}'(\Omega \times (t_1, t_2)).$$

We assume that u and v are continuous in $\bar{\Omega} \times [t_1, t_2]$ and that $u(x,t) \leqslant v(x,t)$ on $\partial\Omega \times (t_1, t_2)$. Then for every $t \in (t_1, t_2)$, and every $R > 0$, noting $\Omega_R = \Omega \cap B_R(0)$, we have the estimate

$$\int_{\Omega_R} (u-v)_+(t) \leqslant \int_{\Omega_{2R}} (u-v)_+(t_1) + C \cdot R^{N-2/(1-\mu)}.$$

In particular, if $u(t_1) \leqslant v(t_1)$, then

$$u \leqslant v \quad \text{in } \bar{\Omega} \times [t_1, t_2].$$

Proof. Let us first assume that φ is smooth and that u, v are also smooth. Then it is clear that using (5.2) and (5.3), the method employed in [11, Proposition 1.4] applies with obvious adaptations. Now, to handle the general case, let us notice that by Corollary 2.8 of [15], any weak solution may be viewed as a local uniform limit of smooth solutions $\{u_k\}$ for the problem associated with a smooth φ_k. Hence we derive the inequality for the u_k and v_k and pass to the limit to get the same for u and v. $\square$

We end this section by the "Radiation Lemma", which was done in the power case in Lemma 2.1 of [11]. Although the adaptations here are easy, we give a proof since we shall refer frequently to this result in the present paper.

Lemma 2.5. *Let $\varphi \in \mathscr{L}^*_{\rm sub}$ and $u \in \mathscr{C}^0(Q; \mathbb{R}_+ \cup \{+\infty\})$ such that, noting $\Omega = \{(x,t) \mid u(x,t) < \infty\}$,*

$$u_t = \Delta\varphi(u) \quad in\ \mathfrak{D}'(\Omega).$$

Assume that there exists a point $y \in \mathbb{R}^N$ and a sequence $t_n \to 0$ such that for any $r > 0$,

$$\int_{B_r(y)} u(x, t_n)\, dx \underset{t_n \to 0}{\to} +\infty.$$

Then for any $c > 0$, noting $v_{c\delta_y}$ the fundamental solution with mass c at *$x = y$, we have*

$$u \geqslant v_{c\delta_y} \quad in\ \mathbb{R}^N \times (0, \infty).$$

Proof. The proof is almost the same as in the case $\varphi(u) = u^m$ [11, Lemma 2.1]: Let y be as above, $c>0$ fixed and $0<r<R$. We choose $\tau>0$ such that

$$\int_{B_r(y)} u(x,\tau)\,dx \geqslant 2c,$$

and since the integral may be infinite, let $f\in\mathscr{C}^0(B_R(y))$ with compact support in $B_r(y)$ such that

$$0\leqslant f(x)<u(x,\tau), \quad \int_{B_r(y)} f(x)\,dx = c,$$

which is always possible if τ is small enough. Let $v_{c,r}$ be the solution of the following problem:

$$\begin{cases} \partial_t v_{c,r} = \Delta\varphi(v_{c,r}) & \text{in } B_R(y)\times(\tau,T), \\ v_{c,r}(x,t)=0 & \text{on } \partial B_R(y)\times(\tau,T), \\ v_{c,r}(x,\tau)=f(x) & \text{in } B_R(y). \end{cases}$$

Assuming temporarily that u and $v_{c,r}$ are positive and smooth in $\Omega_R = \Omega\cap\{B_R(y)\times(\tau,T)\}$, we may compare them exactly as in [11, Lemma 2.1]:

$$0 = v_{c,r}(x,t)<u(x,t) \quad \text{on } \partial B_R\times(\tau,T),$$

while $v_{c,r}(\tau)<u(\tau)$ in B_R by definition of $v_{c,r}$. Thus, $v_{c,r}$ and u remain strictly ordered up to a time

$$t_0 = \max\{t\in(\tau,T)\,|\,u(x,t)>v_{c,r}(x,t)\ \forall x\in B_R(y)\}>\tau.$$

If t_0 is finite, then there exists a $x_0\in B_R(y)$ such that $u(x_0,t_0)=v_{c,r}(x_0,t_0)$ and clearly $x_0\notin\partial\Omega$ because u is infinite on $\partial\Omega$. Thus there exists a small cylinder

$$\bar{B}_\eta(x_0)\times[t_1,t_0]\subset\Omega\cap\{B_R(y)\times(\tau,t_0]\}.$$

In this small cylinder, by continuity and positivity of both u and $v_{c,r}$, we see that it is impossible for them to touch (because $u-v_{c,r}>0$ in this set, satisfies a nondegenerate equation), thus t_0 is infinite, which proves that

$$u>v_{c,r} \quad \text{in } B_R(y)\times(\tau,T).$$

Now using Corollary 2.8 of [15], we know that in the general case, u and $v_{c,r}$ may be approximated by smooth, positive solutions in Ω_R, (and a smooth φ), so that after passage to the limit,

$$u\geqslant v_{c,r} \quad \text{in } B_R(y)\times(\tau,T).$$

The proof ends as in [11] by letting $r\to0, \tau\to0$: $v_{c,r}$ converges to the fundamental solution with mass c at y, but in $B_R(y)\times(0,T)$. Finally $R\to+\infty$ gives by uniqueness:

$$u(x,t)\geqslant v_{c\delta_y}(x,t) \quad \text{in } \mathbb{R}^N\times(0,T),$$

hence the result since $T>0$ is arbitrary. $\square$

3. The infinite point-source solution

We now define a special kind of Razor Blade, which is the IPSS. This kind of solution, having strong singularities at $x=0$ for any $t>0$ was completely studied in [11] in the case $\varphi(u)=u^m$. We give below a generalization of our results.

Definition 3. An IPSS is a Razor Blade such that for any $t>0$, any $r>0$,

$$\int_{B_r(0)} u(x,t)\,dx = +\infty.$$

3.1. Existence of maximal and minimal IPSS

We will construct below two IPSS, a minimal and a maximal one.

Lemma 3.1. *Let $\varphi \in \mathscr{L}^*_{\mathrm{sub}}$ and for any $c>0$, let v_c be the fundamental solution with mass c placed at $x=0$. Then*

$$\underline{U}_\varphi(x,t) = \lim_{c \nearrow +\infty} v_c(x,t)$$

is an IPSS which has the self-similar form

$$\underline{U}_\varphi(x,t) = \underline{f}(|x|t^{-1/2}), \quad \underline{f} \in \mathscr{C}^0(0,\infty). \tag{3.1}$$

Moreover, for any other IPSS u,

$$u \geqslant \underline{U}_\varphi \quad \text{in } Q.$$

Proof. It is clear that the limit $\underline{U}_\varphi$ exists in $\mathbb{R}^N_* \times [0,\infty)$ by local uniform estimates in this set (see (2.7) with $v_c=0$ outside $x=0$), the limit being monotone. Moreover, $\underline{U}_\varphi$ is continuous in $\mathbb{R}^N_* \times (0,\infty)$, and is not integrable near $x=0$ for any $t>0$ (by the estimates of Theorem 2.1 with $s=0$, when $c \to +\infty$). Thus $\underline{U}_\varphi$ is indeed an IPSS. Now we define the scaling operator

$$T_k u(x,t) = u(kx, k^2 t),$$

which transforms any solution u into another solution $T_k u$. In particular, $T_k v_c$ is another fundamental solution, but the mass is changed:

$$T_k v_c(0) = k^{-N} \delta_0(x).$$

Thus by uniqueness of continuous solutions,

$$v_{ck^{-N}}(x,t) = T_k v_c(x,t) = v_c(kx, k^2 t).$$

Letting c increase to $+\infty$ yields the following invariance for $\underline{U}_\varphi$:

$$\underline{U}_\varphi(x,t) = \underline{U}_\varphi(kx, k^2 t),$$

and taking $k = 1/t^{1/2}$ gives

$$\underline{U}_\varphi(x,t) = \underline{U}_\varphi(xt^{-1/2}, 1) = \underline{f}(|x|t^{-1/2}).$$

Since $\underline{U}_\varphi$ is continuous in $\mathbb{R}^N_* \times (0,\infty)$, it is clear that $\underline{f} \in \mathscr{C}^0(0,\infty)$. Finally, the minimality of $\underline{U}_\varphi$ comes from the Radiation Lemma (Lemma 2.5), with $y = \{0\}$: since u is an IPSS, then for any $c>0$,

$$u \geqslant v_c \quad \text{in } Q,$$

so that letting $c \to +\infty$ gives the result. □

Lemma 3.2. *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$ and for $\varepsilon, c>0$, let us consider the weak solution $v_{\varepsilon,c}$ in Q with initial data*

$$v_{\varepsilon,c}(0) = \begin{cases} c & \text{if } |x| \leqslant \varepsilon, \\ 0 & \text{if } |x| > \varepsilon. \end{cases}$$

Then

$$\bar{U}_\varphi = \lim_{\varepsilon \searrow 0} \lim_{c \nearrow +\infty} v_{\varepsilon,c}$$

is an IPSS which has the self-similar form $\bar{U}_\varphi(x,t) = \bar{f}(|x|t^{-1/2})$, with $\bar{f} \in \mathscr{C}^0(0,\infty)$. Moreover, for any IPSS u, we have

$$u \leqslant \bar{U}_\varphi \quad \text{in } Q.$$

Proof. The existence of $\bar{U}_\varphi$ does not pose any problem, as well as the properties of $\bar{f}$, provided we prove that $\bar{U}_\varphi$ has the self-similar form. This last property will follow from the maximality of $\bar{U}_\varphi$, as we shall see later. Let us first prove that $\bar{U}_\varphi$ is indeed the maximal IPSS.

It is clear (by local uniform estimates) that when $c \to +\infty$, $U_\varepsilon = \lim v_{c,\varepsilon}$ exists and it is a continuous function in $Q_\varepsilon = \{|x| > \varepsilon\} \times [0,\infty)$, such that

$$\partial_t U_\varepsilon = \Delta\varphi(U_\varepsilon) \quad \text{in } \mathfrak{D}'(Q_\varepsilon).$$

Moreover, we claim that for any $y \in B_\varepsilon(x)$,

$$U_\varepsilon(x,t) \geqslant \underline{U}_\varphi(x-y,t) \quad \text{in } Q. \tag{3.2}$$

This claim comes again from the Radiation Lemma (Lemma 2.5), with any $y \in B_\varepsilon(x)$. In fact, we prove that for any $c>0$, $U_\varepsilon(x,t) \geqslant v_c(x-y,t)$ and pass to the limit as $c \to +\infty$. Thus (3.2) proves that $U_\varepsilon \equiv +\infty$ on $B_\varepsilon \times (0,\infty)$, and that $U_\varepsilon \in \mathscr{C}^0(Q; \mathbb{R}_+ \cup \{+\infty\})$, that is, U_ε tends to $+\infty$ as $|x| \to \varepsilon$, locally uniformly in time. When ε decreases to zero, we get in the limit,

$$\bar{U}_\varphi(x,t) \geqslant \underline{U}_\varphi(x,t) \quad \text{in } Q,$$

which proves that $\bar{U}_\varphi$ is indeed an IPSS since $\bar{U}_\varphi$ satisfies the equation in $\mathbb{R}^N_* \times (0,\infty)$ and $\bar{U}_\varphi(x,0) = 0$ for $x \neq 0$.

Now for any IPSS u, and any $\varepsilon > 0$, we may compare u with U_ε in Q_ε: we first use Lemma 2.4 with $\Omega = \{|x| > \varepsilon'\}$ and $t_1 = 0$. For any $\tau > 0$ and ε' sufficiently close to ε,

$$U_\varepsilon(x, t+\tau) \geqslant u(x,t) \quad \text{on } |x| = \varepsilon', t \geqslant 0,$$

$$U_\varepsilon(x,\tau) \geqslant u(x,0) = 0 \quad \text{on } |x| \geqslant \varepsilon',$$

so that

$$u(x,t) \leqslant U_\varepsilon(x, t+\tau) \quad \text{in } |x| \geqslant \varepsilon', t \geqslant 0.$$

Letting $\tau \to 0$ and $\varepsilon' \to \varepsilon$ yields

$$u(x,t) \leqslant U_\varepsilon(x,t) \quad \text{in } Q_\varepsilon.$$

Thus when $\varepsilon \to 0$, we find $u \leqslant \bar{U}_\varphi$, which proves that $\bar{U}_\varphi$ is maximal.

Finally, we prove the self-similar form for $\bar{U}_\varphi$: notice that

$$T_k \bar{U}_\varphi(x,t) = \bar{U}_\varphi(kx, k^2 t)$$

is another IPSS, which implies (by maximality of $\bar{U}_\varphi$) that for any $k > 0$,

$$\bar{U}_\varphi(kx, k^2 t) \leqslant \bar{U}_\varphi(x,t).$$

But changing the variables this way: $kx = y$, $k^2 t = s$ yields that for any $k > 0$,

$$\bar{U}_\varphi(y,s) \leqslant \bar{U}_\varphi(y/k, t/k^2),$$

so that $\bar{U}_\varphi$ has the self-similar form

$$\bar{U}_\varphi(x,t) = \bar{f}(|x|t^{-1/2}) = \bar{U}_\varphi(xt^{-1/2}, 1),$$

and continuity of $\bar{f}$ follows from the continuity of $\bar{U}_\varphi$. $\square$

3.2. Uniqueness of the IPSS

The uniqueness of the IPSS was shown in [11] by using the separate-variable form $U_\varphi = t^{1/(1-m)} f(x)$, where f satisfies a well-known elliptic problem which has a unique solution. However, for general φ's, this separate-variable form is not available to us, which implies to find some new argument. In fact the proof is quite simple, but we need before to study the behaviour of $\underline{f}$:

Lemma 3.3. *Let* $\underline{f}$ *be as in Lemma* 3.1. *Then* $\underline{f}$ *is nonincreasing on* $(0, \infty)$, *and*

$$\lim_{\eta\to 0^+} \underline{f}(\eta) = +\infty, \qquad \lim_{\eta\to +\infty} \underline{f}(\eta) = 0.$$

Proof. We will note $\underline{f} = f$. The limit at $+\infty$ is easily shown: we know that $\underline{U}_\varphi$ is continuous up to $t = 0$ for $x \neq 0$ and takes on the zero initial data for $x \neq 0$, so that for $x \neq 0$,

$$f(|x|t^{-1/2}) = \underline{U}_\varphi(x, t) \underset{t\to 0}{\to} 0,$$

which implies that $f \to 0$ when $\eta \to +\infty$.

Step 1. *Monotonicity in the smooth case*. Let us assume that φ and f are smooth. Then we write the equation satisfied by f under the form:

$$[(\varphi \circ f)'\eta^{N-1}]' = -\tfrac{1}{2}\eta^N f'. \tag{3.3}$$

Assume that there exists $\eta \in (0, \infty)$ such that $f'(\eta) > 0$. Then there exists a neighbourhood of η where f' remains positive, so that f is locally increasing. Since f goes to zero at infinity, there exists a point $\eta_0 > 0$ and an $\varepsilon > 0$ such that

$$f'(\eta_0) = 0 \quad \text{and} \quad f' < 0 \quad \text{in }]\eta_0, \eta_0 + \varepsilon[.$$

Thus we integrate (3.3) between η_0 and $\eta_1 \in]\eta_0, \eta_0 + \varepsilon[$:

$$(\varphi \circ f)'\eta^{N-1}\Big|_{\eta_0}^{\eta_1} = \varphi(f(\eta_1))f'(\eta_1)\eta_1^{N-1} = \frac{-1}{2}\int_{\eta_0}^{\eta_1} \eta^N f'(\eta)\, d\eta > 0.$$

But we reach a contradiction since $\varphi' > 0$ and $f'(\eta_1) < 0$. Thus, in the case when φ and f are smooth,

$$f' \leqslant 0 \quad \text{on } (0, \infty).$$

In fact, the exact assumption needed is that $\varphi \in \mathscr{C}^1$, which implies that f is regular where it is positive.

Step 2: *Monotonicity in the general case*. Since φ is not assumed to be smooth, we approximate it by smooth filtration functions φ_k, such that $\varphi_k \to \varphi$ locally uniformly in $(0, \infty)$, and let v_c^k be the fundamental solution with initial data $c\delta_0$ associated with φ_k. Then from Theorem 2.1 follows a local uniform bound of the v_c^k in $\mathbb{R}^N_* \times [0, \infty)$, which is also uniform with respect to k since H_{φ_k} converges to H_φ locally uniformly in $(0, \infty)$. Thus when c increases to $+\infty$, we have a uniform convergence (with respect to $k \in \mathbb{N}$ and (x, t) in compact subsets of $\mathbb{R}^N_* \times [0, \infty)$) of the v_c^k to the minimal IPSS associated with φ_k, noted $\underline{U}_{\varphi_k}$. Since the convergence is uniform with respect to k, we may write

$$\lim_{c\to+\infty}\lim_{k\to+\infty} v_c^k = \lim_{k\to+\infty}\lim_{c\to+\infty} v_c^k,$$

which yields

$$\underline{U}_\varphi = \lim_{k\to+\infty} \underline{U}_{\varphi_k}.$$

By Step 1, we know that $\eta \mapsto \underline{f}_k(\eta) = \underline{U}_{\varphi_k}(\eta, 1)$ is nonincreasing in $(0, \infty)$, thus the same result holds for $\eta \mapsto \underline{f}(\eta) = \underline{U}_\varphi(\eta, 1)$ in the limit.

Step 3. *Limit at* $\eta = 0$. Since $\underline{f}$ is nonincreasing, we know that the following limit exists:

$$\lim_{\eta\searrow 0^+} \underline{f}(\eta) \in [0, \infty].$$

Now this limit cannot be finite because $\underline{U}_\varphi$ is not integrable near $x = 0$ for $t > 0$. Thus

$$\lim_{\eta\to 0} \underline{f}(\eta) = +\infty,$$

which ends the lemma. $\square$

Thanks to this lemma, we can now prove the uniqueness of the IPSS.

Theorem 3.4. *Let* $\varphi \in \mathscr{L}^*_{\text{sub}}$. *Then there exists a unique IPSS noted* U_φ.

Proof. We give below a simple proof which works in the case $N = 1$. We only need to show that $\bar{U}_\varphi = \underline{U}_\varphi$. Since $\underline{U}_\varphi(x,t) = \underline{f}(|x|t^{-1/2})$, let us consider the function

$$W_\varepsilon(x,t) = \underline{f}(|x|t^{-1/2} - \varepsilon) \quad \text{in } \mathscr{P}_\varepsilon = \{|x|t^{-1/2} > \varepsilon\}.$$

It is clear that W_ε is a solution of $u_t = \Delta\varphi(u)$ in $\mathscr{P}_\varepsilon$ which takes on the value $+\infty$ on $\{|x|t^{-1/2} = \varepsilon\}$ (by Lemma 3.3). Thus we are able to compare W_ε with $\bar{U}_\varphi$ in this set. In fact, we shall first compare $W_\varepsilon(t)$ and $\bar{U}_\varphi(t-\tau)$ on

$$Q_{\varepsilon,\tau} = \{|x|t^{-1/2} \geqslant \varepsilon\} \cap \{t > \tau\},$$

in order to avoid the problem at $(x,t) = (0,0)$. Then

$$W_\varepsilon(x,t) = +\infty \geqslant \bar{U}_\varphi(x, t-\tau) \quad \text{on } \{|x|t^{-1/2} = \varepsilon\} \cap \{t > \tau\},$$

$$W_\varepsilon(x,\tau) \geqslant \bar{U}_\varphi(x,0) = 0 \quad \text{on } \{|x|t^{-1/2} = \varepsilon\}.$$

However, $Q_{\varepsilon,\tau}$ is not regular so that we may consider a sequence $\{Q_n\}$ of regular open sets converging to $Q_{\varepsilon,\tau}$. If n is big enough, Q_n is sufficiently close to $Q_{\varepsilon,\tau}$ and thus by continuity of the solutions, we have

$$W_\varepsilon(x,t) \geqslant \bar{U}_\varphi(x,t-\tau) \quad \text{on } \partial Q_n.$$

By a variant of Lemma 2.4 (see [11, Proposition 1.4], with obvious adaptations concerning the surface integrals), we obtain

$$W_\varepsilon(x,t) \geqslant \bar{U}_\varphi(x,t-\tau) \quad \text{in } Q_n.$$

Letting successively $n \to +\infty, \tau \to 0$ yields

$$W_\varepsilon(x,t) \geqslant \bar{U}_\varphi(x,t) \quad \text{in } \mathscr{P}_\varepsilon,$$

and finally $\varepsilon \to 0$ gives the result:

$$\underline{U}_\varphi \geqslant \bar{U}_\varphi \quad \text{in } \mathbb{R}^N_* \times (0,\infty).$$

Hence $\underline{U}_\varphi \equiv \bar{U}_\varphi$, which establishes uniqueness of the IPSS. □

3.3. Solutions having only strong singularities

We end this section with the construction of another kind of Razor Blade, starting with nonzero initial data for $x = 0$, which will be useful later on. This lemma comes from our work with Juan-Luis Vazquez [11] in the case $\varphi(u) = u^m$. In fact, the arguments we used there hold also for our φ's since they only rely upon a basic comparison lemma (Lemma 2.4), and the Radiation Lemma 2.5. We thus omit the proofs.

Lemma 3.5. *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$ and $u_0 \in \mathscr{C}^0(\mathbb{R}^N_*)$, $u_0 \geqslant 0$. Then there exists a unique $u \in \mathscr{C}^0(\mathbb{R}^N \times [0,\infty); \mathbb{R}_+ \cup \{+\infty\})$ such that*

$$u_t = \Delta\varphi(u) + u_0(x) \otimes \delta_0(t) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N_* \times [0,\infty)),$$

and u has strong singularities on $x = 0$ for any $t > 0$.

Proof. See [11]. □

4. The axial trace result

We now turn to the question of giving a sense to the value of singular solutions on the t-axis. In fact, we shall see that either we are in presence of a strong singularity at any time $t > 0$ (the IPSS), or we can define the trace of u on $|x| = 0$ by the second member created in the equation.

Theorem 4.1. *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$ and u be a Razor Blade. Then either $u \equiv U_\varphi$ (the IPSS), or there exists a $\tau \equiv \tau(u) \in]0, \infty]$ such that*

$$u \in L^1_{\text{loc}}([0,\tau) \times \mathbb{R}^N), \quad \textit{while for any } t > \tau, \quad u(t) \notin L^1_{\text{loc}}(\mathbb{R}^N).$$

In particular, $u(t) \geqslant U_\varphi(t-\tau)$ for any $t > \tau$.

Proof. If, there exists a sequence $t_n \to 0$ such that $\int_{B_r(0)} u(x, t_n) \to +\infty$ for any $r > 0$, then we prove as in the Radiation Lemma [11] that for any $c > 0$, $u \geqslant v_c$ where v_c is the fundamental solution with mass $c > 0$ concentrated at $x = 0$. Letting $c \to +\infty$ yields that $u \geqslant U_\varphi$, which proves that u is an IPSS. Thus by uniqueness of the IPSS,

$$u \equiv U_\varphi.$$

Now, if $u \not\equiv U_\varphi$, we can define τ by

$$\tau = \sup\{t > 0 \mid u(x,t) \in L^1_{\text{loc}}(\mathbb{R}^N)\}$$

and it is clear that $u(t)$ is not integrable for any $t > \tau$. Moreover, the lower estimate for such a solution comes also from the Radiation Lemma (Lemma 2.5). $\square$

We shall note $\mathscr{S} = [\tau, \infty)$ and $\mathscr{R} = \mathbb{R}_+ \backslash \mathscr{S}$, according to the notations employed in [11]. We now prove the axial trace result on $\mathscr{R} = [0, \tau)$:

Theorem 4.2. *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$ and u be a singular solution and $\tau = \tau(u)$ as above. Then there exist a measure λ in $\mathscr{M}^+([0,\tau))$, such that*

$$u_t = \Delta\varphi(u) + \delta_0(x) \otimes \lambda(t) \quad \textit{in } \mathfrak{D}'(\mathbb{R}^N \times [0,\tau)).$$

Proof. Since $u \in L^1_{\text{loc}}([0,\tau) \times \mathbb{R}^N)$, we may consider the distribution

$$T = u_t - \Delta\varphi(u) \in \mathfrak{D}'([0,\tau) \times \mathbb{R}^N).$$

Since $u_t = \Delta\varphi(u)$ in Q_*, and $u(0) = 0$, the support of T is included in $\{0\} \times [0, \tau]$, and thus T has necessarily the form

$$T = \delta_0(x) \otimes \lambda(t) + \sum_{i=1}^N \partial_i \delta_0(x) \otimes \sigma_i(t),$$

where λ and $\{\sigma_i\}_{i=1..N}$ are Radon measures on $[0, \tau)$. Indeed, if we use a test function and look at the integration by parts formula in an exterior ball $|x| > r$, we see that T is of order zero in t and order 1 in x. Moreover, if we consider a test function of the form $\psi_\varepsilon(x) = \psi(x/\varepsilon)$, where $0 \leqslant \psi \leqslant 1$ has compact support in $B_1(0)$, we find that

$$\int u(t)\psi_\varepsilon - \frac{1}{\varepsilon^2}\int_0^t \int \varphi(u)(\Delta\psi)(x/\varepsilon)\,dx\,dt$$
$$= \int_0^t \psi(0,s)\,d\lambda(s) + \frac{1}{\varepsilon}\sum_{i=1}^N \int_0^t \partial_i\psi(0,s)\,d\sigma_i(s).$$

But

$$\frac{1}{\varepsilon^2}\left|\int_0^t \int \varphi(u)\Delta\psi(x/\varepsilon)\right| \leqslant \frac{1}{\varepsilon^2}\int_0^t \int_{|x|<\varepsilon} \varphi(u)(x,t)\,dx\,dt$$
$$\leqslant \varepsilon^{N-2}\int_0^t \int_{|x|<1} \varphi(u)(y,t)\,dy\,dt,$$

which remains bounded as $\varepsilon \to 0$ for $N \geqslant 2$. Since

$$\int u(t)\psi_\varepsilon \quad \text{and} \quad \int_0^t \psi(0,s)\,d\lambda(s)$$

remain also bounded as $\varepsilon \to 0$, we deduce that necessarily, $\sigma_i = 0$ for any $i = 1 \dots N$, and the result is proved. Moreover, it is clear that the measure λ is nonnegative, by taking suitable test functions. $\square$

Definition 4. Let $\varphi \in \mathscr{L}^*_{\text{sub}}$. For any Razor Blade u, we thus define the axial trace of u by

$$\operatorname{tr}_{|x|=0}(u) = (\lambda, \tau).$$

Remark. The initial value at $t = 0$ of any Razor Blade may be $c\delta_0$, for some $c \geqslant 0$ (or $c = +\infty$ for the IPSS). In fact, this initial value is taken into account in $\lambda(0)$, hence if $\tau < \infty$, we may either write $u_t = \Delta\varphi(u) + \delta_0(x) \otimes \lambda(t)$ in $\mathfrak{D}'(\mathbb{R}^N \times [0, \tau))$, or

$$\begin{cases} u_t = \Delta\varphi(u) + \delta_0(x) \otimes \lambda(t) & \text{in } \mathfrak{D}'(\mathbb{R}^N \times (0, \tau)), \\ u(0) = \lambda(0) & \text{in the sense of weak convergence} \\ & \quad \text{in measure.} \end{cases}$$

5. Existence of razor blades

We begin with the following estimates for the problem with a forcing term.

Proposition 5.1. *Let $\varphi \in \mathscr{L}^*_{\text{sub}}$ smooth and $u_0 \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, $f \in \mathscr{C}_0^\infty(Q)$ both nonnegative. Then if u is a smooth solution of the following problem*:

$$\begin{cases} u_t - \Delta\varphi(u) = f & \text{in } \mathfrak{D}'(\mathbb{R}^N \times (0, \infty)), \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

we have the following bounds for u:

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^N} u(x,t)\,dx;\ \int_0^t\int_{\mathbb{R}^N} u(x,t)\,dx\,dt \leqslant ||f||_{L^1(Q)} + ||u_0||_{L^1(\mathbb{R}^N)}.$$

Moreover, for any $p<\mu+2/N$, *there exists a* $C(p,\gamma,\mu,N,R,T,u_0,f)>0$ *such that*

$$\int_0^T\int_{B_R} u(x,t)^p \leqslant C(p,\gamma,\mu,N,R,T,||u_0||_{L^1(B_{2R})},||f||_{L^1(B_{2R}\times(0,T))}).$$

Proof. *Step* 1. We denote by $C(f,u_0)$ the quantity

$$C(f,u_0) = ||f||_{L^1(Q)} + ||u_0||_{L^1(\mathbb{R}^N)},$$

and following [18], we consider a test function ψ^θ, where $\psi\in\mathscr{C}_0^\infty(\Omega)$, $0\leqslant\psi\leqslant 1$ and $\psi=1$ on B_R, $\psi=0$ outside B_{2R}. Then clearly,

$$\int_{\mathbb{R}^N} u(x,t)\psi^\theta(x)\,dx - \int_0^t\int_{\mathbb{R}^N}\varphi(u)\Delta(\psi^\theta) \leqslant C(f,u_0), \tag{5.1}$$

and we can estimate (see [18, Lemma 3.1])

$$\int_0^t\int_{\mathbb{R}^N}\varphi(u)|\Delta(\psi^\theta)| \leqslant C(\psi)\left[\int_0^t\int_{\mathbb{R}^N} u\psi^\theta\right]^\mu, \tag{5.2}$$

with

$$C(\psi)\leqslant c_0R^{N(1-\mu)-2} = C(R). \tag{5.3}$$

Then writing $X(t)=\int_0^t\int_{\mathbb{R}^N} u\psi$, we obtain the inequality

$$\frac{d}{dt}X(t)\leqslant C(f,u_0) + C(R)X^\mu \leqslant C(R)(X+1) + C(f,u_0).$$

Then using Grönwall's lemma, we get

$$X(t)\leqslant\{C(f,u_0)+C(R)\}e^{C(R)t},$$

which implies that

$$\int_0^t\int_{B_R} u(x,t)\,dx\,dt \leqslant \{C(f,u_0) + c_0R^{N(1-\mu)-2}\}e^{c_0R^{N(1-\mu)-2}t}.$$

Since $\mu>m_c$, $N(1-\mu)-2$ is negative so that letting R increase to infinity yields

$$\int_0^t\int_{\mathbb{R}^N} u(x,t)\,dx\,dt \leqslant C(f,u_0).$$

Then we come back to (5.1), using (5.2) and (5.3) to get

$$\int_{B_R} u(x,t)\,dx \leqslant C(R)\left[\int_0^t\int_{\mathbb{R}^N} u\varphi\right]^{\mu} + C(f,u_0),$$

hence the result after letting $R\to+\infty$, since $C(R)\to 0$ and $u\in L^1(Q)$.

Step 2: We now turn to the local estimate, which is done in [?] in the case of the equation $u_t = \Delta u^m - u^p$, $m>1$. The presence of the reaction term f instead of u^p does not change the arguments of the proof, so that we shorten some passages: let us take $\phi\in\mathscr{C}_0^\infty(B_{2R})$ with $0\leqslant\phi\leqslant$, $\phi=1$ on B_R, $|\nabla\phi|\leqslant CR^{-1}$ and let us multiply the equation by $u^\alpha(1+u^\alpha)^{-1}\phi^2$. After integrating by parts, we obtain

$$\begin{aligned}
&\int_{B_{2R}}\phi^2\int_0^{u(x,t)}\frac{s^\alpha}{1+s^\alpha}\,ds\,dx\\
&\quad+\frac{4\alpha}{(1-\alpha)^2}\int_0^t\int_{B_{2R}}\phi^2\tfrac{u^{2\alpha}}{(1+u^\alpha)^2}\,\varphi'(u)|\nabla u^{\frac{1-\alpha}{2}}|^2\,dx\,d\tau\\
&\quad\leqslant-\int_0^t\int_{B_{2R}}2\phi\frac{u^\alpha}{(1+u^\alpha)}\,\nabla\varphi(u)\nabla\phi\,dx\,d\tau\\
&\qquad+\int_0^t\int_{B_{2R}}\phi^2\frac{u^\alpha}{(1+u)^\alpha}f\,dx\,d\tau+\int_{B_{2R}}\phi^2\frac{u^\alpha}{(1+u)^\alpha}u_0\,dx.
\end{aligned}$$

By the Schwarz inequality,

$$\begin{aligned}
&\left|\int_0^t\int_{B_{2R}}2\phi\,\frac{u^\alpha}{(1+u^\alpha)}\nabla\varphi(u)\nabla\phi\,dx\,d\tau\right|\\
&\quad\leqslant\frac{2\alpha}{(1-\alpha)^2}\int_0^t\int_{B_{2R}}\phi^2\,\frac{u^{2\alpha}}{(1+u^\alpha)^2}\,\varphi'(u)|\nabla u^{\frac{1-\alpha}{2}}|^2\,dx\,d\tau\\
&\qquad+\;C(\alpha)\int_0^t\int_{B_{2R}}u^{1+\alpha}\varphi'(u)|\nabla\phi|^2\;dx\,d\tau,
\end{aligned}$$

and thus we get

$$\begin{aligned}
&\int_0^t\int_{B_{2R}}\phi^2\frac{u^{2\alpha}}{(1+u^\alpha)^2}\varphi'(u)|\nabla u^{\frac{1-\alpha}{2}}|^2\,dx\,d\tau\\
&\quad\leqslant C(\alpha,R,||f||_{L^1(B_{2R}\times(0,T))},||u_0||_{L^1(B_{2R})})\\
&\qquad\times\left\{1+\int_0^t\int_{B_{2R}}u^{1+\alpha}\varphi'(u)|\nabla\phi|^2\;dx\,d\tau\right\}.
\end{aligned}$$

Using (2.6), we obtain

$$\begin{aligned}
&C_1'\int_0^t\int_{B_{2R}}\phi^2\frac{u^{2\alpha}}{(1+u^\alpha)^2}u^{\gamma-1}|\nabla u^{\frac{1-\alpha}{2}}|^2\,dx\,d\tau\\
&\quad\leqslant C(\alpha,R,f,u_0)\left\{1+C_2'\int_0^t\int_{B_{2R}}u^{\mu+\alpha}|\nabla\phi|^2\;dx\,d\tau\right\}.
\end{aligned}$$

Now if we put $u_1 = \max\{u; 1\}$, and using $u^{\gamma-1}|\nabla u^{\frac{1-\alpha}{2}}|^2 = C(\alpha,\gamma)|\nabla u^{\frac{m_- - \alpha}{2}}|^2$, we obtain for some $C = C(\alpha, \gamma, \mu, R, ||f||_{L^1(B_{2R}\times(0,T))}, ||u_0||_{L^1(B_{2R})})$ that

$$\int_0^t \int_{B_{2R}} \phi^2 |\nabla u_1^{\frac{m_1-\alpha}{2}}|^2 \, dx\, d\tau \leqslant C\left\{1 + \int_0^t \int_{B_{2R}} \phi^2 |\nabla u_1^{\frac{\mu-\alpha}{2}}|^2 \, dx\, d\tau\right\}.$$

Using Sobolev's imbedding when $N \geqslant 3$, we have

$$\int_0^t \int_{B_{2R}} u_1^{m_1+2/N-\alpha} \phi^2 \ dx\, d\tau \leqslant C\left\{1 + \int_0^t \int_{B_{2R}} u^{\mu+\alpha} |\nabla \phi|^2 \ dx\, d\tau\right\}.$$

Finally, we take $\phi = \psi^b$, $\psi \in \mathscr{C}_0^\infty(B_{2R})$, $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on B_R where

$$b \geqslant \frac{\gamma + 2/N - \alpha}{\gamma - \mu + 2/N - 2\alpha}$$

to get

$$\int_0^t \int_{B_{2R}} u_1^{m_1+2/N-\alpha} \psi^{2b} \ dx\, d\tau$$
$$\leqslant C\left\{1 + \left(\int_0^t \int_{B_{2R}} \psi^{2b} u^{\gamma+2/N-\alpha} \, dx\, d\tau\right)^{\frac{\mu+\alpha}{\gamma+2/N-\alpha}}\right\},$$

which in turn gives the result, with $C(\alpha, \gamma, \mu, N, R, T, \ ||f||_{L^1(B_{2R}\times(0,T))}, ||u_0||_{L^1(B_{2R})})$. Note that the assumption

$$m_c < \gamma \leqslant \mu < 1$$

is necessary to conclude by this method. Indeed, this implies that for some $\alpha > 0$ small enough,

$$\mu + \alpha < \gamma + 2/N - \alpha,$$

so that the last Hölder's inequality is possible. In the case $N = 1$ or $N = 2$, the Sobolev's imbedding is a little different, but the result covers all the range $0 < \gamma \leqslant \mu < 1$. $\square$

Now we can prove our existence result:

Theorem 5.2. *Let $\varphi \in \mathscr{L}^*_{\mathrm{sub}}$, $\tau > 0$ and $\lambda \geqslant 0$, $\lambda \not\equiv 0$ a Radon measure on $[0, \tau)$. Then there exists a* Razor Blade *u with axial trace (λ, τ). This means that*

$$u_t - \Delta\varphi(u) = \lambda(t) \otimes \delta_0(x) \quad in\ \mathfrak{D}'(\mathbb{R}^N \times [0, \tau)),$$

and u has strong singularities at $x = 0$ for $t > \tau$:

$$u(x,t) \geqslant U_\varphi(x, t-\tau) \quad for\ t > \tau. \tag{5.4}$$

Proof. Let us consider an approximation $\rho_n \in \mathscr{C}_0^\infty(Q)$, $\rho_n \geqslant 0$, such that $\rho_n \rightharpoonup \lambda \otimes \delta_0$ in $\mathbb{R}^N \times [0,\tau)$. Let also φ_k be a sequence of smooth filtration functions converging to φ locally uniformly, satisfying also (2.2). Then from standard parabolic results, we know there exists a smooth solution $u_{k,n}$ of

$$u_t - \Delta\varphi_k(u) = \rho_n(x,t) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times [0,\tau)).$$

By Corollary 2.8 of [15] (with obvious adaptations to handle the regular right-hand side of the equation), we know that the sequence $u_{k,n}$ converges locally uniformly to some u_n which is a solution of

$$u_t - \Delta\varphi(u) = \rho_n \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times [0,\tau)).$$

Now we write the equation in the weak form: for any $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^N \times [0,\tau))$,

$$-\int_0^t \int_{\mathbb{R}^N} u_n \psi_t - \int_0^t \int_{\mathbb{R}^N} \varphi(u_n)\Delta\psi = \int_0^t \int_{\mathbb{R}^N} \rho_n(x,t)\psi(x,t)\,dx\,dt.$$

Outside the axis $|x| = 0$, we have local uniform bounds for the u_n, so that (up to extraction), u_n converge pointwise to some u. Moreover, the estimates of Proposition 5.1 remain valid for u_n, which give bounds for u_n in $L^p_{\mathrm{loc}}(\mathbb{R}^N \times [0,\tau))$, for any $p < \mu + 2/N$. Since $\mu + 2/N > 1$, we have equi-integrability of the u_n and also of $\varphi(u_n)$: there exists a $p \in (1, \mu + 2/N)$ such that for any measurable set $E \subset \mathbb{R}^N \times [0,t]$,

$$\int\int_E u_n \leqslant |E|^{1-1/p} \left(\int\int_E u_n^p \right)^{1/p} \leqslant C|E|^{1-1/p},$$

$$\int\int_E \varphi(u_n) \leqslant C_2 \int\int_E (u_n)^\mu \leqslant C_2 |E|^{1-\mu/p} \left(\int\int_E u_n^p \right)^{\mu/p}.$$

Thus by equi-integrability and pointwise convergence, we obtain in the limit:

$$-\int_0^t \int_{\mathbb{R}^N} u\psi_t - \int_0^t \int_{\mathbb{R}^N} \varphi(u)\Delta\psi = \int_0^t \psi(0,t)\,d\lambda(t),$$

that is,

$$u_t - \Delta\varphi(u) = \lambda(t) \otimes \delta_0(x) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N \times [0,\tau)).$$

Then the L^∞_{loc} estimates outside $|x| = 0$ yield that $u(\tau)$ is defined and it is continuous in $\mathbb{R}^N_*$. Now we initialize the problem at time $t = \tau$ by the solution v constructed in Lemma 3.5, which starts with $u_0(x) = u(x,\tau)$ for $x \neq 0$ and possesses a strong singularity at $x = 0$ for any $t > 0$. Then it is clear that the solution

$$w(x,t) = \begin{cases} u(x,t) & \text{if } t \in [0,\tau], \\ v(x,t) & \text{if } t > \tau \end{cases}$$

is the Razor Blade we were looking for. The lower bound (5.4) follows from Lemma 3.5. □

6. Uniqueness of Razor Blades

We begin with a local comparison result, adapted from [15]

Lemma 6.1. *Let Ω be a regular bounded open subset of $\mathbb{R}^N$, $t_1 < t_2$ two real numbers and $\varphi \in \mathscr{L}_{\text{sub}}$. Assume that $u, v \in \mathscr{C}^0(\bar{\Omega} \times [t_1, t_2]; \mathbb{R}_+\{+\infty\}) \cap L^1(\Omega \times (t_1, t_2))$, $u(t_1) \leqslant v(t_1)$ and*

$$u_t - \Delta\varphi(u) = v_t - \Delta\varphi(v) \quad \text{in } \mathfrak{D}'(\Omega \times (t_1, t_2)).$$

Then

$$u(x,t) \leqslant v(x,t) \quad \text{in } \bar{\Omega} \times [t_1, t_2].$$

Proof. The proof follows [14, Lemma 2.3]. The modification here is that we allow infinite values for the functions u and v in $D = \Omega \times (t_1, t_2)$. In fact, this is not a problem since

$$A(x,t) = \begin{cases} \dfrac{\varphi(u) - \varphi(v)}{u - v} & \text{if } u(x,t) \neq v(x,t), \\ \varphi'(u) & \text{if } u(x,t) = v(x,t) \end{cases}$$

remains bounded when u or v is infinite, because (2.6) implies that $\varphi'(u) \to 0$ as $u \to \infty$. Thus, we may use the dual method which we summarize here, referring to [14, Lemma 2.3] for the details.

Let

$$A_k = \frac{|\varphi(u) - \varphi(v)|}{\varepsilon_k + |u - v|} + \varepsilon_k, \quad \text{where } \varepsilon_k \searrow 0,$$

and for $s \in (t_1, t_2)$, $h \in \mathscr{C}_0^\infty(\Omega)$, $h \geqslant 0$, let η_k be the solution of the problem

$$\begin{cases} \partial_t \eta + A_k \Delta \eta = 0 & \text{in } D, \\ \eta = 0 & \text{on } \partial\Omega \times (t_1, s), \\ \eta(T) = h & \text{in } \Omega. \end{cases}$$

Note that A_k is bounded away from 0 and $+\infty$ and continuous, so that the existence of a smooth $\eta_k \geqslant 0$ is clear. Then using η_k as test function, we get

$$\begin{aligned} &\int (u - v)(s,x) h(x)\, dx \\ &\leqslant \int_{t_1}^{s} \int_{\Omega} (u - v)(A - A_k) \Delta \eta_k \\ &\leqslant \left[\int_{t_1}^{s} \int_{\Omega} A_k |\Delta \eta_k|^2 \right]^{1/2} \left[\int_{t_1}^{s} \int_{\Omega} \frac{(u - v)^2}{A_k} (A - A_k)^2 \right]^{1/2} \\ &\leqslant C \varepsilon_k \left[\int_{t_1}^{s} \int_{\Omega} |\varphi(u) - \varphi(v)| \right]^{1/2}. \end{aligned}$$

Indeed, $\int_{t_1}^{s}\int_{\Omega} A_k|\Delta\eta_k|^2 \leqslant C(h)$ and $(A - A_k) = \varepsilon_k A/(\varepsilon_k + |u - v|)$, so that

$$\frac{(u-v)^2}{A_k}(A - A_k)^2 \leqslant \varepsilon_k |\varphi(u) - \varphi(v)|.$$

Thus when $k \to \infty$, we get

$$\int (u - v)(s, x)h(x)\,dx \leqslant 0,$$

and since $h \geqslant 0$ is arbitrary, we deduce that $u(s) \leqslant v(s)$ in Ω. Since s also is arbitrary, the result holds. □

From this result, we construct a minimal solution to our problem, and deduce uniqueness of Razor Blades.

Theorem 6.2. *Let $m_c < m < 1$ and (λ, τ) be any axial trace. Then there exists a unique singular solution u with axial trace (λ, τ).*

Proof. If $\tau = 0$, then we know that $u \equiv U_\varphi$, the IPSS. If not, $\tau > 0$ and

$$u \in L^1_{\mathrm{loc}}([0, \tau); L^1_{\mathrm{loc}}(\mathbb{R}^N)),$$

while u is not integrable near $x = 0$ for any time greater than τ.

Step 1: *Construction of a minimal solution*. For $0 < s < \tau$ and $R > 0$, let us consider the solution $v_{s,R}$ of the following problem:

$$\begin{cases} v_t = \Delta\varphi(v) + \lambda(t) \otimes \delta_0(x) & \text{in } B_R \times (0, s), \\ v(x, 0) = 0 & \text{in } B_R, \\ v(x, t) = 0 & \text{on } \partial B_R \times (0, s). \end{cases}$$

It is easy to construct such a solution, with an approximation of $\lambda \otimes \delta_0$ which is smooth. Indeed, if $f_n \rightharpoonup \lambda \otimes \delta_0$, the associated solution v_n (which exists by classical theory) will be bounded by the solution w_n in the whole space with the same forcing term. Hence we deduce that the v_n will be uniformly bounded in $L^p(B_R \times (0, s))$, for any $1 < p < \mu + 2/N$, and also locally uniformly bounded in

$$Q^*_{R,s} = (\overline{B_R} \backslash \{0\}) \times [0, s].$$

Thus after some extraction, we know that v_n converges to some solution $v_{R,s}$ which is continuous in $Q^*_{R,s}$ and will take the lateral and initial data in the classical sense.

Then Lemma 6.1 applies, which gives

$$v_{R,s} \leqslant u \quad \text{in } B_R \times (0, s).$$

Now we let s increase to τ and R to $+\infty$, which yields a solution v of the problem in $\mathbb{R}^N \times (0, \tau)$, and for any other solution u,

$$v \leqslant u \quad \text{in } \mathbb{R}^N \times (0, \tau),$$

that is, v is minimal in the set of Razor Blades. Note that v may be constructed independently of any other solution since $v_{R,s}$ is unique.

Step 2: *Uniqueness on* $(0,\tau)$. On this interval, we use exactly the method of [15, Lemma 3.26], which is even easier in our case since we have a minimal solution v. Let u be any Razor Blade with axial trace (λ,τ) and v the minimal solution constructed on $\mathbb{R}^N\times(0,\tau)$ with right-hand side $\lambda\otimes\delta_0$ on $(0,\tau)$. Starting from

$$\partial_t(u-v)=\Delta(\varphi(u)-\varphi(v)),$$

if $\psi\in\mathscr{C}_0^\infty(\mathbb{R}^N)$ and $w=u-v\geqslant 0$, we obtain

$$\frac{d}{dt}\int\psi(x)w(x)\leqslant CC_\mu(\psi)\left(\int\psi(x)w(x,t)\,dx\right)^\mu + CC_\beta(\psi)\left(\int\psi(x)w(x,t)\,dx\right)^\beta.$$

We refer to [15, Lemma 3.26] for the proof that integrating this ODE and using the fact that

$$h(x,t)=\int_0^t\{\varphi(u)-\varphi(v)\}\geqslant 0$$

is subharmonic in x, one gets $h\equiv 0$ in $\mathbb{R}^N\times(0,\tau)$. Hence

$$u(x,t)=v(x,t)\quad\text{in }\mathbb{R}^N\times[0,\tau),$$

which proves that u is uniquely determined on $[0,\tau)$, and also $u(x,\tau)$ for $x\neq 0$ (by continuity).

Step 3: *Uniqueness on* (τ,∞). On this interval, u has strong singularities at $x=0$ for any t. But the value of u at $t=\tau$ is uniquely determined by λ as *Step* 2 showed. Thus we may apply Lemma 3.5: on (τ,∞), $u(t)$ is uniquely determined by $u(\tau)$, thus u is uniquely determined on $\mathbb{R}^N\times[0,\infty)$ by the axial trace (λ,τ). $\square$

7. Self-similarity and asymptotic behaviour

In this section, we consider the power-case $\varphi(u)=u^m$, with $m_c<m<1$ and want to investigate the different asymptotic behaviours depending on the axial trace (λ,τ). We begin with a study of the case $\tau<\infty$, i.e., $\mathscr{S}\neq\emptyset$, which is contained in [11] (for $t>\tau$).

Then in the case $\tau=+\infty$, we consider (as a model) the case of power right-hand side $\lambda(t)=t^\sigma$ for some $\sigma\geqslant 0$. We denote by u_σ the unique solution of

$$u_t=\Delta u^m+t^\sigma\delta_0(x)\quad\text{in }\mathfrak{D}'(\mathbb{R}^N\times[0,\infty)),$$

and we shall see, the asymptotic behaviour differs according to the power $\sigma\geqslant 0$.

7.1. The case $\tau<\infty$

Let us first mention what happens in the case of an axial trace (λ,τ), with finite τ. In this case, u is of VSS type for $t>\tau$, and more precisely, we have

$$u(x,t)\geqslant\left(\frac{C(t-\tau)}{|x|^2}\right)^{\frac{1}{1-m}}=v_{\tau,\infty}.$$

In fact, u may be viewed as an extended continuous solution in $\mathbb{R}^N\times[\tau,\infty)$ with trace $(\{0\},u(\tau))$ at $t=\tau$. Hence the asymptotic behaviours of u are given in [11], for instance:

$$\lim_{t\to+\infty} t^{\frac{1}{1-m}}|u-v_{\tau,\infty}|=0,$$

locally uniformly in $\mathbb{R}^N_*$. This shows that when τ is finite, the long-time behaviour is given by the IPSS, that is, u grows like $t^{\frac{1}{1-m}}$ for $|x|\neq 0$:

Proposition 7.1. *Let u be a singular solution with axial trace $\mathrm{tr}_{|x|=0}(u)=(\lambda,\tau)$, where $\tau<+\infty$. Then locally uniformly in $\mathbb{R}^N_*$, we have*

$$u(x,t)\underset{t\to+\infty}{\in}\mathcal{O}(t^{\frac{1}{1-m}}).$$

More precisely, the following development is locally uniform in $\mathbb{R}^N_*$:

$$u(x,t)=\left(\frac{Ct}{|x|^2}\right)^{\frac{1}{1-m}}+\mathcal{O}(t^{\frac{m}{1-m}}).$$

7.2. The case $0\leqslant\sigma<m/(1-m)$

We shall now investigate the general case $\sigma\geqslant 0$ and find self-similar solutions to see the behaviour near $|x|=0$ and $t=+\infty$.

Proposition 7.2. *If $0\leqslant\sigma\leqslant\frac{m}{1-m}$, the solution u_σ has the self-similar form*

$$u_\sigma(x,t)=t^\alpha f(x/t^\beta),\quad \alpha=\frac{2\sigma+2-N}{N(m-1)+2},\quad \beta=\frac{m-\sigma(1-m)}{N(m-1)+2},\tag{7.1}$$

where $f=f_\sigma>0$ is the unique continuous (with extended values) solution of

$$-\Delta f^m+\alpha f-\beta\eta\cdot\nabla f=\delta_0\quad \text{in } \mathbb{R}^N.\tag{7.2}$$

In particular,

$$u_\sigma(x,t)\underset{|x|\to 0}{\sim} t^{\sigma/m}E_N(x)\quad \text{locally uniformly in time},\tag{7.3}$$

$$u_\sigma(x,t)\underset{t\to+\infty}{\sim} t^{\sigma/m}E_N(x)\quad \text{locally uniformly in } \mathbb{R}^N_*.\tag{7.4}$$

Proof. If we look for solutions under the self-similar form, we get the above representation and we look for solutions with zero initial data for $|x|\neq 0$, thus for $\beta>0$, which requires indeed

$$0\leqslant\sigma<m/(1-m).$$

We refer to Appendix A for the existence of solutions to (7.2), and the behaviour which is given by

$$f(x)\sim E_N(x)^{1/m}\quad\text{near } |x|=0,$$

so that as $|x|\to 0$,

$$u_\sigma(x,t)\sim t^\alpha E_N(xt^{-\beta})^{1/m}\sim t^{\alpha+(N-2)\beta/m}E_N(x).$$

A straightforward computation shows that $\alpha+(N-2)\beta/m=\sigma/m$, which gives

$$u_\sigma(x,t)\sim t^{\sigma/m}E_N(x)^{1/m}\quad\text{as } |x|\to 0.$$

The second behaviour is obtained by a similar computation. □

Remark. When $\beta>1/2$, i.e., $\sigma<(N-2)_+/N$, then $\alpha<0$, but since $f\to 0$ at $+\infty$, the initial data of the self-similar solution is zero at $t=0$, for $|x|\neq 0$. Let us also mention three special cases:

The case $\sigma=0$. Here $\beta=m/\{N(m-1)+2\}$ and $\alpha=(2-N)/\{N(m-1)+2\}$, which leads to a cancellation of the power of t in the asymptotic behaviour. In fact, it is clear that

$$u_0(x,t)\leqslant E_N(x)^{1/m},$$

since $E_N(x)^{1/m}$ is a stationary solution with positive initial data, but more precisely, $\sigma=0$ implies that locally uniformly in $\mathbb{R}^N_*$,

$$u_0(x,t)\underset{t\to+\infty}{\to}E_N(x)^{1/m}.$$

The case $\sigma=(N-2)_+/N$. We have a self-similar solution with $\beta=1/2$ and $\alpha=0$:

$$u_{\frac{(N-2)_+}{N}}(x,t)=f(xt^{-1/2}).$$

The case $\sigma=m/(1-m)$. In this case, we have the separate-variable form

$$u_{\frac{m}{1-m}}(x,t)=t^{\frac{1}{1-m}}f(x),$$

where $\psi(x)=f^m(x\sqrt{1-m})$ satisfies

$$-\Delta\psi+\psi^q=\delta_0,\quad 1<q=1/m<\frac{N}{(N-2)_+}.$$

We know (see [34] and the references therein) that in this range, such a solution exists, is unique and as above, $\psi(x)$ behaves like $E_N(x)$ as $|x|\to 0$ (see also appendix).

7.3. The case $\sigma > m/(1-m)$

We prove now that when $\sigma > m/(1-m)$, the long-time behaviour is given by the IPSS, that is, $u(t)$ always grows like $t^{1/(1-m)}$, which is the greatest power of time one can get (because any singular solution is majorized by the IPSS).

Theorem 7.3. *Let $\sigma > m/(1-m)$. Then the following limit holds*:

$$\lim_{t\to+\infty} t^{\frac{1}{1-m}}|u_\sigma(x,t) - v_\infty(x,t)| = 0,$$

locally uniformly in $\mathbb{R}^N_*$.

Proof. For $k>1$, let us consider the transform $T_k(u_\sigma) = k^{-1/(1-m)}u_\sigma(x,kt)$. Then

$$\partial_t T_k(u_\sigma) - \Delta T_k(u_\sigma)^m = k^{-\frac{m}{1-m}+\sigma}t^\sigma \delta_0.$$

Thus it is clear that when $k\to+\infty$, $T_k(u_\sigma)(x,t)\to v_\infty(x,t)$ locally uniformly in $\mathbb{R}^N_* \times [0,\infty)$. Indeed, since $\sigma > m/(1-m)$, the right-hand side converges to $+\infty\,\delta_0$ as $k\to+\infty$ (this monotone convergence does not pose any problem). In particular, for $t=1$ we have

$$\lim_{k\to+\infty} k^{-\frac{1}{1-m}}u_\sigma(x,k) = \left(\frac{C}{|x|^2}\right)^{\frac{1}{1-m}},$$

locally uniformly in $\mathbb{R}^N_*$. Finally taking $t=k$ yields the result. $\square$

Remark. When $\sigma = m/(1-m)$, $u(t)$ also grows like $t^{1/(1-m)}$, but the IPSS does not give the long-time behaviour. In fact, in this case one has

$$\lim_{t\to+\infty} t^{\frac{1}{1-m}}u_\sigma(x,t) = f(x),$$

since equality always holds for any t.

8. Extensions and comments

We would like to end this paper with some extensions of the results in this paper, and make some comments.

8.1. General filtration

The study of Razor Blades in the case of super-linear φ's can be handled by similar techniques. However, the main difference in that case is that localization of strong singularities is not possible. This is also the case for the heat equation (i.e., $\varphi(u) = u$): the limit of fundamental solutions blows up everywhere. This fact may be seen as a consequence of the absence of local back-and-fro control of mass. We refer to our forthcoming paper [10] concerning this problem. Some interesting questions also arise when we consider the general filtration equation $u_t = \Delta\varphi(u)$, with

$$m_c < \mu \leqslant \frac{u\varphi'(u)}{\varphi(u)} \leqslant \nu < +\infty.$$

Under this assumptions, we may encounter both effects of slow and fast diffusion. Some results are also contained in [10], but we hope to give a complete description what happens in this case in future works.

8.2. Right-hand side measures

We can easily adapt existence and uniqueness techniques to get the following result:

Theorem 8.1. *Let $\lambda_s \geqslant 0$ be a measure which is singular with respect to the Lebesgue measure in $\mathbb{R}^N \times (0, \infty)$. Then there exists a unique solution u of the following problem*:

$$\begin{cases} u_t - \Delta\varphi(u) = \lambda_s & \text{in } \mathfrak{D}'(\mathbb{R}^N \times (0, \infty)), \\ u(x,0) = 0 & \text{in } \mathbb{R}^N \backslash \mathrm{supp}(\mu_s). \end{cases}$$

Proof. We use the same approximation scheme, with right-hand side $f_n \rightharpoonup \lambda_s$. Uniform bounds in L^p_{loc} for any $1 < p < \mu + 2/N$ come from Proposition 5.1, and since the support of λ_s is closed, of zero measure, we get local uniform outside $\mathrm{supp}(\lambda_s)$, so that almost everywhere convergence holds (up to extraction). Thus, we may pass to the limit in the problem and get a solution. Uniqueness follows from Lemma 6.1 and construction of a minimal solution, following the proof of Theorem 6.2 (steps 1 and 2). $\square$

It should also be noticed that existence and uniqueness may be drawn for Radon measure as initial data, using similar arguments. However, if the right-hand side has a support of positive measure, then pointwise convergence may not hold on this support, so that convergence of the approximations is not clear.

8.3. Extended theory of filtration

A complete extension of the theory of fast-diffusion equations with strong singularities for general functions φ is performed in. Also, a number of results in the present paper are also valid when φ is not concave (same reference).

9. Uncited references

[1–4,7,28]

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Appendix A. An elliptic problem

We prove here some existence result for the equation

$$-\Delta\psi + \alpha\psi^q - \beta x \cdot \nabla(\psi^q) = \delta_0(x) \quad \text{in } \mathbb{R}^N. \tag{A.1}$$

More precisely, we shall show that given $\beta > 0$ and $\alpha \in \mathbb{R}$, there exists a unique solution ψ continuous in $\mathbb{R}^N_*$. Moreover, such a solution is positive, smooth in $\mathbb{R}^N_*$ and belongs to $L^1(\mathbb{R}^N)$. Finally, ψ has the following behaviour near zero:

$$\psi(x) \underset{|x|\to 0}{\sim} E_N(x). \tag{A.2}$$

A.1. Existence of solutions

Let us first consider the approximate problem:

$$-\Delta\psi + \alpha\psi^q - \beta x \cdot \nabla(\psi^q) = f(x).$$

It is clear that for any nonnegative $f \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, there exists a smooth solution $u > 0$ in $\mathbb{R}^N$, and if we assume that f is radial, decreasing, then u is also. Thus the term $-\beta x \cdot \nabla(\psi^q)$ is nonnegative and by comparison, $0 \leqslant \psi \leqslant \psi^*$, where

$$-\Delta\psi^* + \alpha(\psi^*)^q = f.$$

When we take a sequence $f_n \to \delta_0$ in $\mathfrak{D}'(\mathbb{R}^N)$, we then have uniform estimates of the corresponding ψ_n^* in $L^p(\mathbb{R}^N)$ for any $1 < p < N/(N-2)_+$ (whether α is positive or not). By standard L^∞_{loc} estimates, we can extract a sequence converging locally uniformly, and thus ψ_n also converges in $L^p(\mathbb{R}^N)$ for any p as above. We then take a test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} -\psi_n\Delta\varphi + \alpha\int_{\mathbb{R}^N} \psi_n^q\varphi - \beta\int_{\mathbb{R}^N} x \cdot \nabla(\psi_n^q)\varphi = \int_{\mathbb{R}^N} f_n\varphi,$$

and setting $\psi = \lim \psi_n$, we have convergence of every term except the second which is not clear. We rewrite it as

$$\int_{\mathbb{R}^N} x \cdot \nabla(\psi_n^q)\varphi = \int_{\mathbb{R}^N} \operatorname{div}(x\varphi)\psi_n^q,$$

and since ψ_n^q converges at least in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we may pass to the limit in the equation:

$$\int_{\mathbb{R}^N} -\psi\Delta\varphi + \alpha \int_{\mathbb{R}^N} \psi^q\varphi - \beta \int_{\mathbb{R}^N} x \cdot \nabla(\psi^q)\varphi = \varphi(0).$$

Then $\psi \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $\psi \geqslant 0$ solves the problem

$$-\Delta\psi + \alpha\psi^q - \beta x \cdot \nabla(\psi^q) = \delta_0(x) \quad \text{in } \mathfrak{D}'(\mathbb{R}^N).$$

A.2. Uniqueness, regularity and behaviour near zero

Regularity and uniqueness of such a ψ may be drawn from classical elliptic arguments, but we also can deduce them from our work. Indeed, if ψ is continuous and satisfies (A.1) in the sense of distributions, then setting

$$u(x,t) = t^\alpha f(xt^{-\beta}),$$

with $f = \psi^q$ and α, β as in (7.1), we obtain that u is a singular solution which satisfies

$$u_t - \Delta u^m = t^\sigma \delta_0(x) \quad \text{in } \mathbb{R}^N.$$

Hence, by uniqueness of singular solutions (Theorem 3.4), we have $u \equiv u_\sigma$, which implies that u is smooth in $\mathbb{R}^N_* \times (0,\infty)$, and the same for ψ in $\mathbb{R}^N_*$. Smoothness and positivity of u come from the fact that $\varphi(u) = u^m$ is smooth (see [11,12] for the proofs). Note also that since u is a singular solution with empty singular set, $u(t) \in L^1(\mathbb{R}^N)$ for any $t > 0$, thus $\psi \in L^q(\mathbb{R}^N)$, whatever α. This is important since it shows that the term $-\beta \int_{\mathbb{R}^N} x \cdot \nabla(\psi^q)$ has an important effect in the equation as $|x| \to \infty$ (the Green kernel is not integrable at infinity).

We finally show the asymptotic behaviour of ψ near $|x| = 0$. Noting $f = +\alpha\psi^q - \beta x \cdot \nabla(\psi^q) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$-\Delta\psi + f = \delta_0 \quad \text{in } \mathfrak{D}'(\mathbb{R}^N).$$

The following method is well-known and can be found in [34] for instance: let $\bar{\psi} = \psi - E_N$, where E_N is the Green kernel in $\mathbb{R}^N$, then

$$-\Delta\bar{\psi} + f = 0 \quad \text{in } \mathfrak{D}'(\mathbb{R}^N),$$

and $\bar{\psi}$ is regular in $\mathbb{R}^N_*$, so that by radiality, we may write

$$(r^{N-1}\bar{\psi}_r)_r = f(r),$$

and if $0<r<1$, we obtain

$$r_r^{N-1}\bar{\psi}(r)=\int_r^1 f+u(1).$$

Since $f\in L^1_{\text{loc}}$, then $r_r^{N-1}\bar{\psi}(r)$ has a limit c has $r\to 0$. If $c\neq 0$, then $\bar{\psi}$ behaves like $c'r^{2-N}$ near zero, which is impossible because $\Delta\bar{\psi}$ would create a Dirac mass. Thus $c=0$ and we obtain the development near $|x|=0$:

$$\psi(x)=E_N(x)+o(|x|^{2-N}),$$

which proves (A.2).

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